

# Six-body Light-Front Tamm-Dancoff approximation and wave functions for the massive Schwinger model

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## Abstract

The spectrum of the massive Schwinger model in the strong coupling region is obtained by using the light-front Tamm-Dancoff (LFTD) approximation up to including six-body states. We numerically confirm that the two-meson bound state has a negligibly small six-body component. Emphasis is on the usefulness of the information about states (wave functions). It is used for identifying the three-meson bound state among the states below the three-meson threshold. We also show that the two-meson bound state is well described by the wave function of the relative motion.

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## I. INTRODUCTION

In a previous paper [1], we investigated the massive Schwinger model [2,3] with  $SU(2)_f$  in the light-front Tamm-Dancoff (LFTD) approximation [4,5] up to including four-body states. We showed, by examining the wave functions, that the lightest isosinglet state can be regarded as a bound state of two “pions.” This observation naturally led us to the answer to the question raised by Coleman [3] why it is so light. The LFTD approximation has been proved to be one of the most powerful non-perturbative methods to date in the investigation of relativistic bound states, at least in two dimensions, although we have to face the difficult renormalization problem in higher dimensions.

The validity of the LFTD approximation is based on the plausible hope that the sea quark/gluon contributions are small in the light-cone quantization because pair creations/annihilations are suppressed [6]. Typically, the lightest particles are expected to be in the valence states. It is generally true in the models so far investigated. The above mentioned state (the bound state of two “pions”) is an important exception. With this exception, one might think that such a state would have non-negligible many-body components too. It is one of our purposes of this paper to show numerically that it is unlikely by examining the single-flavor model.

We also investigate the three-meson bound state of the single-flavor model. Its existence has been discussed by Coleman [3] by using the bosonization technique. He showed that, in the strong coupling limit with the zero vacuum angle, there exists a stable three-meson bound state and it is unstable when the vacuum angle is non-zero. We look for a candidate which can be interpreted as a bound state of three mesons in our numerical results.

In order to investigate these problems, it is necessary to do LFTD calculations up to including six-body states. Such calculations are very hard without any technical refinement. To make these calculations feasible we have made two points: (1) We take a simple set of basis functions in order to reduce CPU time. A clever choice of basis functions is essential as in quantum chemistry calculations. Note that our choice of basis functions in a finite

domain will be also useful for higher dimensions. Even in higher dimensions the longitudinal momenta  $p_i^+$  of constituents are restricted to a finite domain  $0 \leq p_i^+ \leq P^+$  with  $\sum_i p_i^+ = P^+$ , where  $P^+$  is the total momentum. (2) The three-meson bound state, if it exists, must be in the continuum unless it is lighter than two mesons, and is therefore apparently difficult to find. We can however find a candidate among several states by looking at the wave functions. The points are that a three-meson state must be charge conjugation odd and that below the three-meson threshold, six-body components should be very small except for three-meson bound states. A more detailed discussion is given in Sec. III.

We emphasize that the information about states (wave functions) is very useful. It is used for identifying the three-meson bound state, as is said above. As another example, we introduce the wave function of the relative motion of the two-meson bound state and try to describe the bound state in terms of the wave function. Although the concept of “relative motion” of a relativistic bound state is somewhat awkward, we however find that the two-meson bound state is well described in terms of the wave function of the relative motion, in the sense that a smaller set of basis functions motivated by the concept of the relative motion gives a good approximation. It gives us a qualitative picture of the bound state.

We summarize the results: (1) The masses of the lowest states do not change even if we include six-body states. (2) In particular, the state which can be regarded as a bound state of two mesons has a negligible six-body component. (3) We find a candidate for the bound state of three mesons. (4) The wave function of the relative motion of the two-meson bound state describes the bound state well. We can have a picture that in the strong coupling region it is loosely bound, while in the weak coupling region it is tightly bound, compare to the size of the meson.

The massive Schwinger model [7,8] has been discussed by many authors in the light-cone quantization. Bergknoff [9] did the first LFTD calculations. Mo and Perry [10] refined his calculations by the use of basis functions. Their calculations include only up to four-body states. Eller, Pauli, and Brodsky [11,12] discussed the massless and the massive Schwinger

models in the discretized light cone quantization (DLCQ). Our work is based especially on the papers by Bergknoff, and Mo and Perry. We try to keep our notation as close as possible to that of our previous paper [1].

In Sec. II, we present basic facts and formulas on the massive Schwinger model to make this paper self-contained. The model is quantized on the light cone and the Tamm-Dancoff truncation is made up to including six-body states. The wave functions are expanded in terms of a new set of basis functions. The numerical results are shown in Sec. III. We identify two-meson and three-meson bound states. The two-meson bound state is shown to have a negligibly small six-body component. The three-meson bound state is charge conjugation odd and has a large six-body component compare to those of other states below the three-meson threshold. In Sec. IV, we introduce a meson operator which (approximately) creates a meson from the vacuum. By using the meson operator, we also introduce the wave function of the relative motion. Sec. V is devoted to discussions.

## II. FORMULATION

### A. Definition of the model

The massive Schwinger model [2,3] is two-dimensional QED with a massive fermion. It is not exactly solvable in contrast to the massless one [7,8]. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi, \quad (2.1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . In two dimensions, the coupling constant  $e$  has mass dimension. It is therefore useful to measure all dimensionful quantities in units of  $e/\sqrt{\pi}$ . We hereafter set  $e/\sqrt{\pi} = 1$ . Strong couplings correspond to small fermion masses.

In the light-cone gauge ( $A^+ = 0$ ), the only independent variable is  $\psi_R$  in the light-cone quantization.  $A^-$  and  $\psi_L$  are expressed in terms of  $\psi_R$  as follows:

$$A^- = \sqrt{\pi} \frac{1}{(i\partial_-)^2} j^+, \quad (2.2)$$

$$\psi_L = \frac{m}{\sqrt{2}} \frac{1}{i\partial_-} \psi_R, \quad (2.3)$$

with  $j^+(x^-) = \sqrt{2} : \psi_R^\dagger(x^-) \psi_R(x^-) :$ . We use the principal value prescription for  $(i\partial_-)^{-1}$  and  $(i\partial_-)^{-2}$  as in Refs. [1,10].

Eliminating  $A^-$  and  $\psi_L$  by using (2.2) and (2.3), one obtains the light-cone Hamiltonian  $P^-$ ,

$$\begin{aligned} P^- &= P_{free}^- + P_{int}^- , \\ P_{free}^- &= \frac{m^2}{2\sqrt{2}} \int_{-\infty}^{\infty} dx^- \psi_R^\dagger(x^-) \frac{1}{i\partial_-} \psi_R(x^-) , \\ P_{int}^- &= \frac{\pi}{2} \int_{-\infty}^{\infty} dx^- j^+(x^-) \frac{1}{(i\partial_-)^2} j^+(x^-) . \end{aligned} \quad (2.4)$$

We expand  $\psi_R$  in terms of the creation and annihilation operators,

$$\psi_R(x^-) = \frac{1}{2^{1/4}} \int_0^\infty \frac{dk^+}{(2\pi)\sqrt{k^+}} \left[ b(k^+) e^{-ik^+ x^-} + d^\dagger(k^+) e^{ik^+ x^-} \right] , \quad (2.5)$$

where  $b(k^+)$  and  $d(k^+)$  satisfy the following anti-commutation relations,

$$\{b(k^+), b^\dagger(l^+)\} = \{d(k^+), d^\dagger(l^+)\} = (2\pi) k^+ \delta(k^+ - l^+) , \quad (2.6)$$

derived from  $\{\psi_R(x^-), \psi_R^\dagger(y^-)\} = (1/\sqrt{2})\delta(x^- - y^-)$ . One may express  $P^-$  entirely in terms of  $b(k^+)$  and  $d(k^+)$  (and their Hermitian conjugates). We refer the reader to Ref. [10] for the explicit form.

We work in a truncated Fock space in which a state with total light-cone momentum  $P^+ = \mathcal{P}$  is expressed as

$$\begin{aligned} |\psi\rangle_{\mathcal{P}} &= |2\rangle_{\mathcal{P}} + |4\rangle_{\mathcal{P}} + |6\rangle_{\mathcal{P}} , \\ |2\rangle_{\mathcal{P}} &= \int_0^{\mathcal{P}} \frac{dk_1 dk_2}{\sqrt{(2\pi)^2 k_1 k_2}} \delta(k_1 + k_2 - \mathcal{P}) \psi_2(k_1, k_2) b_1^\dagger d_2^\dagger |0\rangle , \\ |4\rangle_{\mathcal{P}} &= \frac{1}{2} \int_0^{\mathcal{P}} \prod_{i=1}^4 \frac{dk_i}{\sqrt{(2\pi)k_i}} \delta(\sum_{i=1}^4 k_i - \mathcal{P}) \psi_4(k_1, k_2; k_3, k_4) b_1^\dagger b_2^\dagger d_3^\dagger d_4^\dagger |0\rangle , \\ |6\rangle_{\mathcal{P}} &= \frac{1}{3!} \int_0^{\mathcal{P}} \prod_{i=1}^6 \frac{dk_i}{\sqrt{(2\pi)k_i}} \delta(\sum_{i=1}^6 k_i - \mathcal{P}) \psi_6(k_1, k_2, k_3; k_4, k_5, k_6) b_1^\dagger b_2^\dagger b_3^\dagger d_4^\dagger d_5^\dagger d_6^\dagger |0\rangle , \end{aligned} \quad (2.7)$$

where we use the abbreviated notations,  $b_i^\dagger = b^\dagger(k_i)$ ,  $d_i^\dagger = d^\dagger(k_i)$ . We rescale momenta,  $k_i \rightarrow x_i = k_i/\mathcal{P}$  and the wave functions,  $\psi_2(k_1, k_2)$ ,  $\psi_4(k_1, k_2; k_3, k_4)$ , and  $\psi_6(k_1, k_2, k_3; k_4, k_5, k_6)$  are replaced by  $\psi_2(x_1, x_2)$ ,  $\mathcal{P}^{-1}\psi_4(x_1, x_2; x_3, x_4)$ , and  $\mathcal{P}^{-2}\psi_6(x_1, x_2, x_3; x_4, x_5, x_6)$ , respectively.

The wave functions  $\psi_2$ ,  $\psi_4$  and  $\psi_6$  must satisfy the following symmetry properties due to Fermi statistics,

$$\begin{aligned}\psi_4(x_1, x_2; x_3, x_4) &= -\psi_4(x_2, x_1; x_3, x_4) = -\psi_4(x_1, x_2; x_4, x_3) = \psi_4(x_2, x_1; x_4, x_3) , \\ \psi_6(x_1, x_2, x_3; x_4, x_5, x_6) &= -\psi_6(x_2, x_1, x_3; x_4, x_5, x_6) = -\psi_6(x_1, x_3, x_2; x_4, x_5, x_6) \\ &= -\psi_6(x_1, x_2, x_3; x_5, x_4, x_6) = -\psi_6(x_1, x_2, x_3; x_4, x_6, x_5) \text{ etc. } .\end{aligned}\tag{2.8}$$

If we require that this state has a definite property under charge conjugation transformation, we have further conditions on these wave functions,

$$\begin{aligned}\psi_2(x_1, x_2) &= \mp \psi_2(x_2, x_1) , \\ \psi_4(x_1, x_2; x_3, x_4) &= \pm \psi_4(x_3, x_4; x_1, x_2) , \\ \psi_6(x_1, x_2, x_3; x_4, x_5, x_6) &= \mp \psi_6(x_4, x_5, x_6; x_1, x_2, x_3) .\end{aligned}\tag{2.9}$$

The upper/lower sign in (2.9) corresponds to charge conjugation even/odd.

The Einstein-Schrödinger equation  $M^2|\psi\rangle_{\mathcal{P}} = 2P^-P^+|\psi\rangle_{\mathcal{P}}$  leads to a set of complicated eigenvalue equations for the wave functions. It can be converted to a single matrix eigenvalue problem by expanding the wave functions in terms of basis functions, which we discuss in the next subsection.

## B. Basis functions

It has been known that the wave function  $\psi_2(x, 1-x)$  behaves as  $x^\beta$  in the vicinity of  $x = 0$  [9], with  $\beta$  being the solution of the equation  $m^2 - 1 + \pi\beta \cot(\pi\beta) = 0$ . By taking it into account, Mo and Perry concluded that a useful choice of the basis functions for the wave functions is given in terms of Jacobi polynomials,  $P_n^{(\beta, \beta)}$ . In a previous paper [1], we

propose a simpler set of basis functions, essentially equivalent to that of Mo and Perry. We now propose another set of basis functions which leads to a drastic reduction of CPU time. We expand the wave functions as follows.

$$\begin{aligned}
\psi_2(x, 1-x) &= \sum_{k=0}^{N_2} a_k F_k(x, 1-x) , \\
\psi_4(x_1, x_2; x_3, x_4) &= \sum_{\mathbf{k}}^{N_4} b_{\mathbf{k}} G_{\mathbf{k}}(x_1, x_2; x_3, x_4) , \quad \sum_{i=1}^4 x_i = 1 , \\
\psi_6(x_1, x_2, x_3; x_4, x_5, x_6) &= \sum_{\mathbf{K}}^{N_6} c_{\mathbf{K}} H_{\mathbf{K}}(x_1, x_2, x_3; x_4, x_5, x_6) , \quad \sum_{i=1}^6 x_i = 1 ,
\end{aligned} \tag{2.10}$$

where we use the following basis functions:

$$F_k(x, 1-x) = \begin{cases} [x(1-x)]^{\beta+k} \\ [x(1-x)]^{\beta+k} (2x-1) , \end{cases} \tag{2.11}$$

$$G_{\mathbf{k}}(x_1, x_2; x_3, x_4) = (x_1 x_2 x_3 x_4)^{\beta} (x_1 x_2)^{k_1} (x_3 x_4)^{k_2} (x_1 - x_2)(x_3 - x_4)(x_1 + x_2)^{k_3} , \tag{2.12}$$

$$\begin{aligned}
H_{\mathbf{K}}(x_1, x_2, x_3; x_4, x_5, x_6) &= (x_1 x_2 x_3 x_4 x_5 x_6)^{\beta} \\
&\times \left[ (x_1 x_2)^{k_1} (x_1 + x_2)^{k_2} (x_1 - x_2) + (1, 2, 3 \text{ cyclic}) \right] \\
&\times \left[ (x_4 x_5)^{k_3} (x_4 + x_5)^{k_4} (x_4 - x_5) + (4, 5, 6 \text{ cyclic}) \right] (x_1 + x_2 + x_3)^{k_5} .
\end{aligned} \tag{2.13}$$

We abbreviate the upper limits of the sums. In reality,  $N_2 = 2(M_1 + 1)$ , and it means that  $k$  in (2.11) runs from 0 to  $M_1$ . Similarly,  $N_4 = (M_2 + 1)^2(M_3 + 1)$ , i.e.,  $k_1, k_2 = 0, \dots, M_2$ ,  $k_3 = 0, \dots, M_3$  in (2.12) and  $N_6 = (M_4 + 1)^2(M_5 + 1)^2(M_6 + 1)$ , i.e.,  $k_1, k_3 = 1, \dots, M_4 + 1$ ,  $k_2, k_4 = 0, \dots, M_5$ , and  $k_5 = 0, \dots, M_6$  in (2.13). The important point in choosing this set of basis functions is to reduce the number of the factors of the type  $(x_1 + x_2)^k$ . This reduction allows us to express the basis functions in a simple way in the source code. For example,  $G_{\mathbf{k}}(x_1, x_2; x_3, x_4)$  may be written as

$$G_{\mathbf{k}}(x_1, x_2; x_3, x_4) = \sum_{j=0}^{k_3} \sum_{i=0}^3 (-1)^{2-i_1-i_2} \binom{k_3}{j} (x_1 x_2 x_3 x_4)^{\beta} x_1^{N_1} x_2^{N_2} x_3^{N_3} x_4^{N_4} , \tag{2.14}$$

with  $N_1 = k_1 + i_1 + j$ ,  $N_2 = k_1 + 1 - i_1 + k_3 - j$ ,  $N_3 = k_2 + i_2$ ,  $N_4 = k_2 + 1 - i_2$ , and  $i = 2i_1 + i_2$ . (We use a binary number for  $i$ .) Without using this new set of basis functions, six-body LFTD calculations would be much more heavy.

We have explicitly separated the two-body basis functions into charge conjugation eigenfunctions. But we have not done that for four-body and six-body basis functions because it makes the expressions so complicated that the drastic reduction of CPU time cannot be expected. We determine the charge conjugation property of an eigenstate by looking at the two-body state. From our experience, we know that it is a reliable way.

With these expansions, the Einstein-Schrödinger equation becomes a (generalized) matrix eigenvalue problem, which can be solved numerically. Calculations of the matrix elements can be carried out analytically by using the formulas (and their generalizations) collected in an appendix of Ref. [1].

### III. NUMERICAL RESULTS

#### A. Convergence

First of all we have to see how many basis functions are enough to produce reliable results. We set  $m = 0.1$  and gradually increase the number of basis functions. Fig. 1 shows the lowest mass states in the calculation including only two-body basis functions. The lowest state is the meson state. It is charge conjugation odd. Its mass is 1.18160 at  $N_2 = 10$  ( $M_1 = 4$ ). The dashed line indicates the two-meson threshold. Note that the convergence is good enough for  $N_2 = 10$ .

As we increase the number of four-body basis functions, (keeping  $N_2 = 10$ ) a state goes down below the two-meson threshold as shown in Fig. 2. We regard this state as the two-meson bound state. On the other hand, the lowest state, the meson, is little affected by the inclusion of the four-body states. Its mass is 1.18103 at  $N_4 = 80$  ( $M_2 = 3, M_3 = 4$ ), decreased only 0.05%. It is due to a negligibly small four-body component, 0.002%. Note that all the state above the two-meson threshold go down as  $N_4$  increases. We find that  $N_4 = 80$  is enough for the convergence for the lightest two states.

We proceed to the six-body calculations, keeping  $N_2 = 10$  and  $N_4 = 80$  ( $N_2 + N_4 = 90$ ).



As seen in Fig. 3, the convergence is quite good for the states below the three-meson threshold indicated by the dotted line. The three-meson threshold is given by the sum of the meson mass and the mass of the two-meson bound state. Again, the meson is not affected by the inclusion of six-body states. The two-meson bound state does not change either. The mass is 2.30980 at  $N_6 = 36$  ( $M_4 = 2, M_5 = 1, M_6 = 0$ ) which should be compared to 2.31004 in the four-body calculations. The state just below the three-meson threshold can be regarded as the three-meson bound state, as we discuss shortly. It seems that  $N_6 = 36$  ( $N_2 + N_4 + N_6 = 126$ ) is enough for the convergence for the lightest states. In the following, we restrict ourselves to this case.

### B. Two-meson bound state

The second lightest state can be regarded as a bound state of two mesons. Its mass is 2.30980 at  $m = 0.1$ . It has a 72.825% two-body component, 27.174% four-body component and 0.001% six-body component. The ratios change as the fermion mass changes. The smaller gets the fermion mass, the larger four-body component it has. For example, at  $m = 0.01$ , it has 54.408% two-body component, 45.592% four-body component and 0.0004% six-body component. It is important to notice that it has a negligibly small six-body component. From this result we presume that it will not have any many-body components even if we could include higher Fock states. This state is charge conjugation even.

### C. Three-meson bound state

We identify the three-meson bound state by the following criteria; (1) Its mass must be below the three-meson threshold. (2) It must have a large six-body component relative to the other states below the three-meson threshold, at least in the strong coupling region. (3) It must be charge conjugation odd.

The first criterion is a trivial one, and is necessary for distinguishing it from three-meson scattering states.

The second criterion is based on the observation that the meson is almost completely in the valence state in the strong coupling region. As we discuss in detail in the next section, one may consider an (approximate) meson creation operator  $A^\dagger$ . Thus a three-meson state may be represented as  $\sim (A^\dagger)^3|0\rangle$ , which implies that the three-meson bound state has a large six-body component and negligibly small many-body components. Similarly a two-meson state has a negligibly small six-body component.

The third criterion comes from the fact that a meson state is charge conjugation odd.

We find such a state that satisfies all of these criteria. Its mass is 3.39181 at  $m = 0.1$ , well below the threshold, 3.49083. It has a 65.170% two-body component, 34.292% four-body components and 0.538% six-body component. All states near this have smaller six-body components, typically a few hundredths percent or less. For smaller fermion masses, the six-body component of the three-meson bound state become larger. For example, at  $m = 0.001$ , the two-, four-, and six-body components are 42.013%, 55.013%, and 2.974% respectively. It is charge conjugation odd.

One might be surprised that it has a small six-body component, and might suspect that it is a two-meson state. In Sec. V, we will argue that it cannot be regarded as a two-meson state.

#### IV. WAVE FUNCTION FOR THE RELATIVE MOTION OF THE TWO-MESON BOUND STATE

In the previous section, we utilize the information of the wave function to identify the three-meson bound state. It is an outstanding feature that we can get such information. In this section, we consider another example in which the information of the wave function is crucial. We introduce a meson creation operator to have a qualitative picture of the two-meson bound state by considering the wave function of the relative motion.

### A. Meson operator

Let us introduce an operator  $a^\dagger(p)$ ,

$$\begin{aligned} a^\dagger(p) = & \int_0^p \frac{dk}{(2\pi)\sqrt{k(p-k)}} b^\dagger(k) d^\dagger(p-k) \\ & + \int_0^\infty \frac{dk}{(2\pi)\sqrt{k(p+k)}} \left[ b^\dagger(p+k) b(k) - d^\dagger(p+k) d(k) \right] . \end{aligned} \quad (4.1)$$

It is easy to show that it satisfies the following commutation relations,

$$\begin{aligned} [a(p), a^\dagger(q)] &= p\delta(p-q) , \\ [a(p), a(q)] &= [a^\dagger(p), a^\dagger(q)] = 0 , \end{aligned} \quad (4.2)$$

where  $a(p)$  is the Hermitian conjugate to  $a^\dagger(p)$  and annihilates the vacuum,

$$a(p)|0\rangle = 0. \quad (4.3)$$

By using these operators, the Hamiltonian can be written in the following form,

$$P^- = \frac{m^2}{4\pi} \int_0^\infty \frac{dk}{k^2} \left[ b^\dagger(k) b(k) + d^\dagger(k) d(k) \right] + \frac{1}{2} \int_0^\infty \frac{dp}{p^2} a^\dagger(p) a(p) . \quad (4.4)$$

Note that in the massless case  $m = 0$ , the Hamiltonian is diagonal and the eigenstates are the Fock states of  $a^\dagger$ . The operator  $a^\dagger$  is the creation operator of the meson in the (massless) Schwinger model, which is equivalent to a free massive scalar theory,

$$|\text{meson}(m=0)\rangle_{\mathcal{P}} = a^\dagger(\mathcal{P})|0\rangle = \int_0^{\mathcal{P}} \frac{dk}{(2\pi)\sqrt{k(\mathcal{P}-k)}} b^\dagger(k) d^\dagger(\mathcal{P}-k)|0\rangle. \quad (4.5)$$

The meson is structureless in the sense that the wave function has no momentum dependence,  $\psi(k, \mathcal{P}-k) = 1$  for any  $k$ . Compare with (4.6) below.

Once the fermion mass is introduced, the Fock states of  $a^\dagger$  are no longer the eigenstates of the Hamiltonian. The results in the previous section suggest, however, that one may introduce an approximate meson creation operator whose Fock states are approximate eigenstates of the Hamiltonian. We have seen that the wave function of the meson behaves

as  $\psi \sim [x(1-x)]^\beta$  [9] and the higher Fock components are negligible in the strong coupling region. Taking into account these things, we propose the following approximate meson creation operator,

$$A^\dagger(p) = \int_0^p \frac{dk}{(2\pi)\sqrt{k(p-k)}} \psi(k, p-k) b^\dagger(k) d^\dagger(p-k) \\ + \int_0^\infty \frac{dk}{(2\pi)\sqrt{k(p+k)}} \varphi(p+k, k) \left[ b^\dagger(p+k) b(k) - d^\dagger(p+k) d(k) \right], \quad (4.6)$$

where  $\psi$  should be equivalent to  $\psi_2$  for the meson in the previous section and therefore had been known numerically,

$$|\text{meson}(m \neq 0)\rangle_{\mathcal{P}} \approx A^\dagger(\mathcal{P})|0\rangle = \int_0^{\mathcal{P}} \frac{dk}{(2\pi)\sqrt{k(\mathcal{P}-k)}} \psi(k, \mathcal{P}-k) b^\dagger(k) d^\dagger(\mathcal{P}-k) |0\rangle. \quad (4.7)$$

Compare with (2.7). The shape of  $|\psi|^2$  is shown in Fig. 5. On the other hand,  $\varphi$  cannot be determined by looking at the meson state. But it affects two-meson states and can be determined by examining the two-meson bound state, at least in principle.

It is hard to estimate the errors of the approximate *operator*  $A^\dagger(p)$ , though the *state* (4.7) has been shown to be a fairly good approximation. For small fermion masses (small  $\beta$ ), we expect that  $\psi = 1 + \mathcal{O}(\beta)$  and  $\varphi = 1 + \mathcal{O}(\beta)$ , and the errors are expected to be  $\mathcal{O}(\beta)$  because  $A^\dagger(p)$  reduces to  $a^\dagger(p)$  for  $\psi = \varphi \equiv 1$ .

## B. Wave function of the relative motion

Let us attempt to describe the two-meson bound state by introducing the wave function of the relative motion. Such a description is based on the *assumption* that it is a *two-meson* state, i.e., that the two mesons in the bound state would not be distorted too much. The assumption is justified *a posteriori* in the strong coupling region.

Under this assumption, the two-meson bound state may be written as

$$|\text{two-meson}\rangle_{\mathcal{P}} \equiv \frac{1}{\sqrt{2}} \int_0^{\mathcal{P}} \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \delta(p_1 + p_2 - \mathcal{P}) \Phi(p_1, p_2) A^\dagger(p_1) A^\dagger(p_2) |0\rangle \\ = \int_0^{\mathcal{P}} \frac{dk}{(2\pi)\sqrt{k(\mathcal{P}-k)}} \Psi_2(k, \mathcal{P}-k) b^\dagger(k) d^\dagger(\mathcal{P}-k) |0\rangle \quad (4.8)$$

$$+ \frac{1}{2} \int_0^\infty \prod_{i=1}^4 \frac{dk_i}{\sqrt{(2\pi)k_i}} \delta(\sum_{i=1}^4 k_i - \mathcal{P}) \Psi_4(k_1, k_2; k_3, k_4) b_1^\dagger b_2^\dagger d_3^\dagger d_4^\dagger |0\rangle ,$$

where we have substituted (4.6). The wave functions  $\Psi_2$  and  $\Psi_4$  are expressed in terms of  $\psi$  and  $\varphi$  in the following way,

$$\Psi_2(k_1, k_2) = \frac{1}{\sqrt{2}} \int_0^{k_1} \frac{dq}{\sqrt{q(\mathcal{P}-q)}} \Phi(q, \mathcal{P}-q) \varphi(k_1, k_1-q) \psi(k_2, k_1-q) \quad (4.9)$$

$$- \frac{1}{\sqrt{2}} \int_0^{k_2} \frac{dq}{\sqrt{q(\mathcal{P}-q)}} \Phi(q, \mathcal{P}-q) \varphi(k_2, k_2-q) \psi(k_1, k_2-q) ,$$

$$\begin{aligned} \Psi_4(k_1, k_2; k_3, k_4) = & - \frac{\Phi(k_1 + k_3, k_2 + k_4)}{\sqrt{2(k_1 + k_3)(k_2 + k_4)}} \psi(k_1, k_3) \psi(k_2, k_4) \\ & + \frac{\Phi(k_1 + k_4, k_2 + k_3)}{\sqrt{2(k_1 + k_4)(k_2 + k_3)}} \psi(k_1, k_4) \psi(k_2, k_3) . \end{aligned} \quad (4.10)$$

The wave function  $\Phi$  is that of the relative motion of the two mesons in the two-meson bound state.

The meson operator (4.6) does not exactly satisfy the same commutation relations as (4.2), but only approximately. Thus  $\Phi(p_1, p_2)$  does not need to be symmetric under the exchange of  $p_1$  and  $p_2$ . Nevertheless we regard it as being symmetric throughout this paper. Note that the ansatz (4.8) is consistent with the charge conjugation symmetry, that is,  $\Psi_2(k_1, k_2)$  is antisymmetric in  $k_1$  and  $k_2$ ,  $\Psi_4(k_1, k_2; k_3, k_4)$  is symmetric under the exchange of  $(k_1, k_2)$  and  $(k_3, k_4)$ .

It is interesting to note that this ansatz drastically reduces the degrees of freedom in the functional space. Due to the assumption that the meson wave function would not be distorted too much in the bound state, only one degree of freedom, i.e., the relative motion of the mesons, comes in. It seems natural to expand  $\Phi(x_1, x_2)$  (symmetric in  $x_1$  and  $x_2$ ) as

$$\Phi(x_1, x_2) = \sum_{l=0}^N B_l [x_1 x_2]^{l+\frac{1}{2}} , \quad x_1 + x_2 = 1 , \quad (4.11)$$

where  $B_l$  is a coefficient to be determined numerically. Taking into account the fact that  $\psi(x_1, x_2)$  is well approximated by  $a_0(x_1 x_2)^\beta$ , with  $a_0$  being the normalization constant,  $a_0 = [B(2\beta + 1, 2\beta + 1)]^{-1/2}$  ( $B$  is a Beta function), one may consider the following basis function expansions,

$$\Psi_2(x, 1-x) = \sum_{l=0}^N A_l [x(1-x)]^{\beta+l} (2x-1) , \quad (4.12)$$

$$\begin{aligned} \Psi_4(x_1, x_2, x_3, x_4) = & \sum_{l=1}^N \frac{B_l a_0^2}{\sqrt{2}} (x_1 x_2 x_3 x_4)^\beta \\ & \times \left\{ -[(x_1 + x_3)(x_2 + x_4)]^l + [(x_1 + x_4)(x_2 + x_3)]^l \right\} , \end{aligned} \quad (4.13)$$

where  $A_l$  is another coefficient to be determined numerically. Note that this set of basis functions is much simpler than the original one (2.12).

We calculate the mass and the wave functions (i.e., the coefficients  $A_l$  and  $B_l$ ) of the two-meson bound state by using this set of basis functions with  $N = 7$ . The mass is calculated as 2.04180 at  $m = 0.01$ . This result is surprisingly good for this small set of basis functions. It is even better than the result of our full-set calculations,  $m = 2.05612$ . This is because of the factors like  $[(x_1 + x_3)(x_2 + x_4)]^l$  in (4.13), which are suitable for expressing the relative motion of the two mesons. It is therefore expected that this set of basis functions is good only for the two-meson bound state.

The squared wave functions,  $|\Phi(x, 1-x)|^2$ , for various values of the fermion mass are shown in Fig. 6. From this, we have an intuitive picture that the mesons are loosely bounded for small fermion masses, while they are close to each other for large fermion masses. This behavior has a simple physical interpretation: As is seen from Fig. 5, the meson is tightly bounded for small fermion masses. Therefore the very weak Van der Waals force between the two mesons causes a loosely bounded two-meson state. For large fermion masses, on the other hand, the meson has a broad shape. Therefore the charge distribution over a wide region keeps the two mesons close to each other by the Coulomb force.

It is possible to check quantitatively how good the assumption (4.8) is. By inspection, we find that the wave function  $\varphi$  may be written as follows,

$$\varphi(x_1, x_2) = b_0 \left( \frac{x_2}{x_1} \right)^\beta , \quad (4.14)$$

where the constant  $b_0$  is very close to 1 for small fermion masses. This is consistent with the massless limit (4.1), in which  $\beta = 0$ ,  $b_0 = 1$ , and  $\varphi = 1$ . Given the form of  $\varphi$  (4.14) one may express  $\Psi_2$  in terms of  $\Phi$ ,  $\psi$ , and  $\varphi$  as

$$\begin{aligned}\Psi_2(x_1, x_2) = & \frac{1}{\sqrt{2}} \int_0^{x_1} \frac{dy}{\sqrt{y(1-y)}} \Phi(y, 1-y) \varphi(x_1, x_1-y) \psi(x_1-y, x_2) \\ & - \frac{1}{\sqrt{2}} \int_0^{x_2} \frac{dy}{\sqrt{y(1-y)}} \Phi(y, 1-y) \varphi(x_2, x_2-y) \psi(x_1, x_2-y) .\end{aligned}\quad (4.15)$$

By substituting  $\psi$ , (4.14), and (4.11), we obtain the following expression for  $\Psi_2$  in terms of  $B_l$ ,

$$\begin{aligned}\Psi_2(x_1, x_2) = & \frac{a_0 b_0}{\sqrt{2}} (x_1 x_2)^\beta (x_1 - x_2) \sum_{l=0}^N B_l \sum_{m=0}^l (-1)^m \binom{l}{m} B(l+m+1, 2\beta+1) \\ & \times \sum_{k=0}^{\lfloor \frac{l+m}{2} \rfloor} (-1)^k \binom{l+m-k}{k} (x_1 x_2)^k ,\end{aligned}\quad (4.16)$$

where we used the formula,

$$\sum_{l=0}^n x_1^l x_2^{n-l} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \binom{n-m}{m} (x_1 x_2)^m , \text{ for } x_1 + x_2 = 1 . \quad (4.17)$$

This should be compared with (4.12). Fig. 7 shows the wave function  $\Psi_2$  calculated by the direct diagonalization and by eq. (4.16). (The coefficients  $B_1, \dots, B_N$  are obtained by the diagonalization. It is necessary to take into account the normalization condition,

$$\int_0^1 dx |\Phi(x, 1-x)|^2 = 1 , \quad (4.18)$$

to obtain  $B_0$ . Although  $b_0$  still remains undetermined, we simply put  $b_0 = 1$  for the comparison.) The agreement measures the validity of the concept of the “relative motion” of the two mesons inside the two-meson bound state.

## V. DISCUSSIONS

By using a simpler set of basis functions, we have obtained the mass spectrum of the massive Schwinger model in the LFTD approximation. We have confirmed that the two-meson bound state has a negligibly small six-body component and found a candidate for the three-meson bound state. We emphasize that the information on the wave functions is very useful and is used for identifying the three-meson bound state. It is also used for

investigating the two-meson bound state. We introduce an (approximate) meson creation operator and the concept of the relative motion of the two mesons. This description gives an intuitive, qualitative picture of the two-meson bound state and motivates a very simple set of basis functions.

The candidate for the three-meson bound state has a small six-body component compared to the two-body and four-body components, even in the strong coupling region. One might suspect that it is a four-body state, not a six-body state. Actually, the corresponding state appears below the threshold in the four-body calculations. Nevertheless, we think that it is the three-meson bound state for the following reasons. (1) In the massless theory, there exists only a free scalar particle in the physical spectrum. But because the creation operator of the meson, if expressed in terms of the creation and annihilation operators of the fermion and the antifermion as in (4.1), contains the annihilation operators, even a pure (free) three-meson state is not a pure six-body state. It is thus not so strange even if it has a small six-body component in the massive case. (2) The state is charge conjugation odd. A two-meson state should be charge conjugation even. It is hard to imagine that a two-meson state can be charge conjugation odd, if we rely on the description in terms of the meson. The description may be justified in the strong coupling region because the unperturbed (massless) theory has only the meson and the perturbation is small. It is natural to have a picture that the perturbation causes weak interactions between the mesons to form bound states.

Unfortunately, we do not know why the six-body component of the three-meson bound state is so small. It is an outcome of complex non-perturbative effects. An analysis similar to that of Sec. IV B may reveal how the three-meson bound state looks like but will not explain the smallness of the six-body component. At this moment, we have to be content with showing that it *is* the three-meson bound state.

In the strong coupling region, the two- and three-meson bound states appear above the threshold. We have the prejudice that despite our numerical results, they are, in reality, still bound. Probably they are just below the threshold in this region and approach the threshold



in the massless limit. We think that the reason why they do not appear to be bound is due to the limitation of our variational calculations and numerical errors. If one takes it for granted that they are really bound, one may estimate the errors in the calculations.

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## FIGURES

FIG. 1. Two-body Tamm-Dancoff approximation. The lightest states are shown with the total number of basis functions. The fermion mass is  $m = 0.1$ . The lowest state is the meson. All the other states are “spurious.”

FIG. 2. Four-body Tamm-Dancoff approximation. The lightest states are shown with the total number of basis functions. The fermion mass is  $m = 0.1$ . The lowest state does not change at all. The second lowest state goes down below the two-meson threshold (the dashed line). It is the two-meson bound state.

FIG. 3. Six-body Tamm-Dancoff approximation. The lightest states are shown with the total number of basis functions. The fermion mass is  $m = 0.1$ . The lowest two states do not change at all. The state shown by the line with triangle points is the candidate for the three-meson bound state. The three-meson threshold is indicated by the dotted line.

FIG. 4. Fermion mass dependence of the mass eigenvalues. The dashed and dotted lines stand for the two-meson and three-meson thresholds respectively.

FIG. 5. Squared wave functions for the meson,  $|\psi(x, 1 - x)|^2$ , are shown for various values of the fermion mass. For small masses the wave function (in the momentum space) has little  $x$  dependence, implying that the meson is a compact object, while for large masses it is localized around  $x = 1/2$ , implying that the meson has a broad shape, and that the fermion and the antifermion are bound loosely.

FIG. 6. Squared wave functions for the relative motion of the two mesons in the two-meson bound state,  $|\Phi(x, 1 - x)|^2$ , are shown for various values of the fermion mass. For small masses the wave function (in the momentum space) has a sharp peak at  $x = 1/2$ , implying that the mesons are bound loosely, while for large masses it has a round shape, implying that the mesons are very close to each other.

FIG. 7. Consistency check for the ansatz. The dashed line is the wave function  $\Psi_2(x, 1 - x)$  obtained by the direct diagonalization. The solid line is the wave function  $\Psi_2(x, 1 - x)$  constructed from  $\Phi$  and  $\Psi_4$  by using the ansatz (4.8). The fermion mass is set  $m = 0.01$ . The agreement is quite good. It shows that our assumption for the two-meson bound state is quantitatively justified *a posteriori*.

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